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Exact Time Dependent Solutions of Black-Scholes Equation by Lie's Similarity Transformation Method

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Abstract: Exact solutions of the Black-Scholes partial differential equation is found by Lie's similarity transformation method. There are four integration constants. Various cases of the parameter dependence are also studied .

.**Keywords:** Similarity Transformation, Parameter, Infinitesimal, Extensions,

I. Introduction

Black-Scholes mathematical model for option pricing on European stock is a single linear partial differential equation of second order with only one dependent variable v and two independent variables x and t ,in addition two constants γ and σ .

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + \gamma\, x \frac{\partial v}{\partial x} + \frac{\partial v}{\partial t} - \gamma\, v \ = 0\,, \qquad (1.01)$$

where v is the price of the option that depends on the stock price x and time t , σ a constant is the volatility of the stock, and γ another constant is the period.

The boundary condition on the option price depends on the at the maturity time T so that ,

$$v(x, T)= x - c \qquad (1.02)$$

Where x is the stock price and c is the strike price. The caller advised to purchase the option at time T at the strike price c so able to get a profit (x−c) , when x > c . Means ,if stock price x is below the strike price c then the caller not advised to buy the option as that leads to a loss .So

$$v(x, T) =0, \text{ when } x \le c. \qquad (1.03)$$

the time t is measured in proportions of a year.

Model helps the price of a European call option on stock that allows no dividends. Same model equation Merton also derived using very different approach from Black and Scholes and so mathematical model (1.01) also called as Black-Scholes-Merton equation for option pricing on a stock. The call option on a stock allows its holder the right to purchase from the seller one share of the underlying stock, at a predetermined price, at or before an expiry date. Since stock market is volatile so market direction never easily predicted, but Black-Scholes model assume that the volatility of the market is constant over time. So model is depending on the concept of hedging and is a tool to avoid risks associated with the volatility of underlying assets and stock options.

In this study the new exact solutions of the Black-Scholes equation (1.01) are found by the method of Lie's Similarity Transformation (LST). Using LST method (1.01) partial differential equation is transformed to a constant coefficient second order ordinary differential equation and that solved by well known method. Obtained two exact new solutions of Black-Scholes equations with five arbitrary integration constants. Few special cases of the constant of (101) is also reported

II. Lie's Similarity Transformtion Method for Partial Differential Equations

Essential details of the Lie's similarity transformation method to reduce the number of independent variables of a partial differential equation (PDE) so as to obtain respective ordinary differential equation (ODE) [06] is the following. Let the given PDE in two independent variables x and t and one dependent variable v be

$$F(x, t, v, v_t, v_x, u_{xx}, \dots) = 0, \qquad (2.01)$$

where $v_t, v_x \dots are$ all partial derivatives of dependent variables $v(x, t)$ with respect to the independent variable t and x respectively.

When we apply a family of one parameter infinitesimal continuous point group transformations,

$$x = x + \epsilon X(x, t, v) + O(\epsilon^2), \qquad (2.02)$$

$$t = t + \epsilon T(x, t, v) + O(\epsilon^2)\,, \qquad (2.03)$$

$$v = v + \epsilon V(x, t, v) + O(\epsilon^2)\,. \qquad (2.04)$$

We get the infinitesimals of the variables u, t and x as U, T, X respectively and ϵ is an infinitesimal parameter. The derivatives of $u(x, t)$ are also transformed as

$$v_x = v_x + \epsilon[V_x] + O(\epsilon^2)\,, \qquad (2.05)$$

$$v_{xx} = v_{xx} + \epsilon[V_{xx}] + O(\epsilon^2), \qquad (2.06)$$

$$v_{tt} = v_{tt} + \epsilon[V_{tt}] + O(\epsilon^2)\,, \qquad (2.07)$$

where $[V_x], [V_{xx}], [V_{tt}]$ are the infinitesimals of the derivatives v_x, v_{xx}, v_{tt} respectively. These are called first and second extensions and that are given by [06],

$$[V_x] = V_x + (V_v - X_x)v_x - X_v v_x^2 - T_x v_t - T_v v_x v_t\,, \qquad (2.08)$$

$$[V_t] = V_t + (V - T_t)v_t - T_v v_t^2 - X_t\, v_x - X_v v_t v_x\,, \qquad (2.09)$$

$$[V_{xx}] = V_{xx} + (2V_{xv} - X_{xx})v_x - T_{xx} v_t + (V_{vv} - 2X_{xv})v_x^2 - X_{vv} v_x^3, \; + (V_v - 2X_x)v_{xx} - 3X_v v_x v_{xx} - 2T_{xv} v_x v_t - T_{vv} v_x^2 v_t -$$

$$-2T_x v_{xt} - T_v v_{xx} v_t - 2T_v v_{xt} v_x, \qquad (2.10)$$

The invariant requirements of given PDE (2.1) under the set of above transformations lead to the invariant surface conditions,

$$T\frac{\partial F}{\partial t} + X\frac{\partial F}{\partial x} + V\frac{\partial F}{\partial v} + [V_x]\frac{\partial F}{\partial v_x} + [V_{xx}]\frac{\partial F}{\partial v_{xx}} = 0. \qquad (2.11)$$

On solving above invariant surface condition (2.11), the infinitesimals X, T, V can be uniquely obtained, that give the similarity group under which the given PDE (2.1) is invariant. The solution of (2.11) are obtained by Legrange's condition,

$$\frac{dt}{T} = \frac{dx}{X} = \frac{du}{U}. \qquad (2.12)$$

This yields,

$$x = x(t, C_1, C_2), \qquad (2.13)$$

and

$$v = v(t, C_1, C_2) \qquad (2.14)$$

where $C_1$and C_2 are arbitrary integration constants and the constant C_1plays the role of an independent variable called the similarity variable *s(x,t)* and C_2 that of a dependent variable called the similarity solution $v(s)$ such that exact solution of given PDE, so that

$$v(x,\ t) \quad = \quad v(s) \tag{2.15}$$

On substituting (2.15) in given PDE (2.1) that reduced to an ordinary differential equation with *s(x, t)* as independent variable and $u(s)$ as dependent variable.

III . Lie's Similarity Transformation of Black-Scholes Equation

The Black-Scholes equation is

$$\frac{1}{2}\sigma^2\, x^2\, v_{xx} + \gamma\, x\, v_x + v_t - \gamma v \;= 0, \tag{3.01}$$

Then the ' invariant surface condition' for (3.01) is

$$\sigma^2\, 2x\, X\, v_{xx} + \frac{1}{2}\sigma^2\, x^2\, [V_{xx}\,] + \gamma\, X v_x + \gamma\, x\, [V_x] + [V_t] - \gamma\, V = 0 \;\; . \tag{3.02}$$

Substitute the expansions of $[V_x\,]$, $[V_t\,]$ and $[V_{xx}\,]$ then equate different partial derivatives of v as zero gives following essential constrained equations :

$$\frac{1}{2}\sigma^2\, V_{xx} + \gamma\, x\, V_x + V_t - \gamma\, V = 0, \tag{3.03}$$

$$\frac{1}{2}\sigma^2(2V_{xv} - X_{xx}) + \gamma\, x\, (V_v - X_x) - X_t + \gamma\, X = 0, \tag{3.04}$$

$$\frac{1}{2}x^2(V_{vv} - 2X_{xu}) - \gamma\, x\, X_v = 0, \tag{3.05}$$

$$\frac{1}{2}\sigma^2 x^2\, T_{xv} + \gamma\, x\, T_x - X_v = 0, \tag{3.06}$$

$$\frac{1}{2}\sigma^2(V_v - 2\, X_x) + \gamma\, x\, X = 0, \tag{3.07}$$

$$-\gamma\, x\, T_x - \frac{1}{2}\sigma^2 T_{xx} - (V_v - T_t) = 0, \tag{3.08}$$

$$T_x = T_v = T_t = X_v = 0. \tag{3.09}$$

On solving above constrained equations , we get

$$X = C_o x, \tag{3.10}$$

$$T = C_1 \;\; , \tag{3.11}$$

$$V = \exp(\gamma t)\, , \tag{3.12}$$

where C_o and C_1 are arbitrary integration constants and $C_1 \neq 0$.

The Lagrange's condition (2.12) yields,

$$\frac{dx}{C_o x} = \frac{dt}{C_1} = \frac{dv}{\exp(\gamma t)} \;\; . \tag{3.13}$$

On solving (3.13) get "similarity solution "of the Black-Scholes equation (3.01) as

$$v\,(x\,,t) \;=\; \frac{1}{\gamma}\, \exp(\gamma\, t) + f(\eta), \tag{3.14}$$

where η(x, t) is the "similarity variable" and is ,

$$\eta = (\log x - \frac{C_o}{C_1} t). \quad (3.15)$$

On substituting (3.14) in the Black-Scholes equation (3.01) ,get the "similarity reduced " ordinary differential equation with constant coefficients with dependent variable as f(η) and independent variable as the similarity variable η of (3.15) ,

$$\frac{1}{2}\sigma^2 \frac{d^2 f}{d\eta^2} + (\gamma - \frac{1}{2}\sigma^2 - \frac{C_o}{C_1})\frac{df}{d\eta} - \gamma f = 0. \quad (3.16)$$

The exact solution of (3.16) is ,

$$f(\eta, t) = C_2 \exp[C_4\eta] + C_3 \exp[C_5\eta], \quad (3.17)$$

where,

$$C_4 = \{ (-\gamma + \frac{1}{2}\rho^2 + \frac{C_o}{C_1}) + \sqrt{[(\gamma - \frac{1}{2}\sigma^2 - \frac{C_o}{C_1})^2 + 2\sigma^2\gamma]} \ \}/\sigma^2, \quad (3.18)$$

and

$$C_5 = \{ (-\gamma + \frac{1}{2}\rho^2 + \frac{C_o}{C_1}) - \sqrt{[(\gamma - \frac{1}{2}\sigma^2 - \frac{C_o}{C_1})^2 + 2\sigma^2\gamma]} \ \}/\sigma^2 , \quad (3.19)$$

where C_o , C_1, C_3, C_4 and C_5 are all arbitrary integration constants , $C_1 \neq 0$.

IV. Exact Solutions of Black-Scholes Equation for Different Values of σ ,γ , and C_o .

The exact solution of Black-Scholes equation (3.01)is obtained by substituting (3.15),(3.17)-(3.19 in (3.14) as

$$v(x, t) = \frac{1}{\gamma}\exp(\gamma t) + C_2\exp[C_4\log(x - \frac{C_o}{C_1}t)] + C_3\exp[C_5\log(x - \frac{C_o}{C_1}t)] \ , \quad (4.01)$$

where C_2 and C_3 are arbitrary integration constants , both cannot be zero together.
Exact solution can be further simplified as,

$$v(x, t) = \frac{\exp(\gamma t)}{\gamma} + \{C_2\, x^{C_4}\exp(\frac{-C_4 C_o}{C_1}t) + C_3 x^{C_5}\exp(\frac{C_5 C_o}{C_1}t)\}, \quad (4.02)$$

where

$$C_4 = \frac{(\frac{\sigma^2}{2} - \gamma + \frac{C_o}{C_1})}{\sigma^2} + \frac{\sqrt{(\frac{\sigma^2}{2} - \gamma + \frac{C_o}{C_1})^2 + 2\sigma^2\gamma}}{\sigma^2} , \quad (4.03)$$

and

$$C_5 = \frac{(\frac{\sigma^2}{2} - \gamma + \frac{C_o}{C_1})}{\sigma^2} - \frac{\sqrt{(\frac{\sigma^2}{2} - \gamma + \frac{C_o}{C_1})^2 + 2\sigma^2\gamma}}{\sigma^2} \quad (4.04)$$

When the arbitrary constant C_o=0 , then the exact solution (4.02)-(4.04) yield second exact solution $v_o(x,t)$of Black-Scholes equation (3.01) as,

$$v_o(x, t) = \frac{\exp(\gamma t)}{\gamma} + \{ C_2 x^{C_6} + C_3 x^{C_7}\}, \quad (4.05)$$

where

$$C_6 = (\frac{\frac{1}{2}\rho^2 - \gamma}{\sigma^2}) + \frac{\sqrt{(\frac{1}{2}\sigma^2 - \gamma)^2 + 2\sigma^2\gamma}}{\sigma^2} \quad \text{and} \quad C_7 = (\frac{\frac{1}{2}\rho^2 - \gamma}{\sigma^2}) - \frac{\sqrt{(\frac{1}{2}\sigma^2 - \gamma)^2 + 2\sigma^2\gamma}}{\sigma^2} . \quad (4.06)$$

V. Conclusion

In the modelling of Black-Scholes equation it is assumed that the stock option can exercised on its expiration or maturity date as in European options. But that does not exist in American options. Another limitation of

Black-Scholes model's possible in real time. Constant risk free interest , no dividends, and no interest earning are also rarely happen in present stock market..

In above study the exact solutions of Black-Scholes model equation (1,01) solved and found exact solutions. Solutions depends on four integration constants C_o , C_1, C_2 , and C_3 , where $C_1 \neq 0$. The above two solutions (4.02) and(4.05) are valid for different values risk interest rate γ and volatility of stock σ. For example , when

$$\gamma = \frac{1}{2}\sigma^2 , \qquad (5.01)$$

and C_o = 0 the exact solution is,

$$v_1(x, t) = \frac{1}{\gamma}\exp(\gamma\, t) + \{C_2\, x + C_3\, x^{-1}\} \qquad (5.02)$$

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